A Counterexample to Conjectures by Sloane and Erdős concerning the Persistence of Numbers

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If the digits of any multi-digit number are multiplied together, another number results. If this process is iterated, eventually a single digit number will be produced. The number of steps that this process takes, before a single digit number is obtained, is referred to as the persistence of the of the original number [5].

Neil Sloane conjectured that for any base b, there is a number $c(b)$ such that the persistence in base b cannot exceed $c(b)$. According to Richard Guy [2], Erdős Pàl has made a similar conjecture regarding the persistence of numbers in which only non-zero digits are considered. No doubt both Sloane and Erdös were assuming fixed, or single, radix systems when making their conjectures. Nonetheless, this assumption is not explicitly stated, and if a fixed radix system is not assumed, then the conjectures are false.

Readers may recall that in factorial base [4] (also referred to as "factorian") integers are represented as the sum of multiples of factorials [1][3]. The right-most digit represents multiples of 1!, the next digit to the left represents multiples of 2! and so on. For small numbers it is convenient simply to indicate the factorial base thus,

$$37_{10} = (1 \times 4!) + (2 \times 3!) + (0 \times 2!) + (1 \times 1!) = 1201_F.$$

With larger numbers, and particularly when referring to individual digits of the number, it is easier to show the meaning of each digit explicitly within the representation; thus

$$a_{n!}b_{(n-1)!} \ldots c_{2!}d_{1!} = a \times n! + b \times (n-1)! + \ldots + c \times 2! + d \times 1!,$$

where a, b, c, and d represent 'digits', and, for example

$$\begin{aligned} 5305305600_{10} &= (11 \times 12!) + (0 \times 11!) + (10 \times 10!), \\ &= 11_{12!}0_{11!}10_{10!}0_{9!}0_{8!}0_{7!}0_{6!}0_{5!}0_{4!}0_{3!}0_{2!}0_{1!}. \end{aligned}$$

Table 1 shows the persistence in factorial base of numbers in the range 0 to 25 together with the iterated path that each number takes before reaching a single digit.

n	n_F	Path			Persistence
0	0	0			0
1	1	1			0
2	10	10	0		1
3	11	11	1		1
4	20	20	0		1
5	21	21	10	0	2
6	100	100	0		1
7	101	101	0		1
8	110	110	0		1
9	111	111	1		1
10	120	120	0		1
11	121	121	10	0	2
12	200	200	0		1
13	201	201	0		1
14	210	210	0		1
15	211	220	10	0	2
16	220	220	0		1
17	221	221	20	0	2
18	300	300	0		1
19	301	301	0		1
20	310	310	0		1
21	311	311	11	1	2
22	320	320	0		1
23	321	321	100	0	2
24	1000	1000	0		1

Table 1: The persistence of numbers $\leq 4!$

Lemma 1. *No even number has a persistence greater than* 1*. That is, if we let* $P(n)$ *represent the persistence of* n*, then* $n \equiv 0 \bmod 2 \Rightarrow P(n) \leq 1$.

Proof. If $n \equiv 0 \bmod 2$ and $n > 0$, then we can write the factorial base representation as

$$n = a_{x!}b_{(x-1)!}\ldots c_{2!}d_{1!}.$$

Each of the digit terms represents a multiple of 2!, and therefore of 2, with the exception of the rightmost digit d, which must be 0. The product of the digits of n is therefore 0, making $P(n) = 1$. Finally, $P(0) = 0 \leq 1$. □

Lemma 2. *If the factorial representation of* n *contains an even digit, then* $P(n) \leq 2$.

Proof. If any of the digits of n is 0 then $P(n) = 1$. If none of the digits is 0, but at least one of the digits is even, then the factorial base representation of their product (m) will end in a final 0. $P(m) \leq 1$, by Lemma 1, which implies that $P(n) = P(m) + 1 \leq 2$. □

Lemma 3. *If* $n > 2$ *and* $P(n) > 2$ *then* $n \equiv 0 \bmod 3$.

Proof. From Lemma 2, n contains no even digit. The factorial base representation is therefore of the form

$$a_{x!}b_{(x-1)!}\dots 1_{2!}1_{1!}.$$

Each of the digit terms represents a multiple of 3!, and therefore of 3, except for the two rightmost digits which together sum to 3. Thus, $n \equiv 0 \bmod 3$. □

Lemma 4. *It is possible to find a number in factorial base of arbitrarily large persistence. That is,*

$$\forall p\colon\ p > 1, \qquad \exists n\colon\ P(n) = p.$$

Proof. The proof is by construction. Calculate

$$k = (n \times n!) + (1 \times (n-1)!) + (1 \times (n-2)!) + \dots + (1 \times 2!) + (1 \times 1!),$$

the factorial base representation of which is

$$n_{n!}1_{(n-1)!}1_{(n-2)!}\dots 1_{2!}1_{1!}.$$

The product of the digits of k is equal to n. Furthermore, $P(k) = P(n) + 1$ since it will take a single step to transform k into n, and $P(n)$ steps to reach a single digit. Induction on $P(n)$ together with the fact that $P(2) = 1$, completes the proof. □

If n is the smallest number with persistence p, it is not necessarily the case that a number constructed as k above will be the smallest number with persistence $p + 1$. Construction from $5_{10} = 21_F$ shows that $P(633_{10}) = P(51111_F) = 3$, and this is indeed the smallest integer with a persistence equal to 3. However, although by Lemma 4 we know that $P(633_{633!}1_{632!}\dots 1_{2!}1_{1!}) = 4$, this is far from being the smallest number with a persistence equal to 4; that accolade instead belongs to $443155013_{10} = 11_{11!}1_{10!}1_{9!}1_{8!}7_{7!}1_{6!}1_{5!}3_{4!}3_{3!}1_{2!}1_{1!}$.

Given $P(n) = p$, our method of construction does however provide an upper bound on the smallest number with persistence $p + 1$.

Lemma 5. *There is no upper bound on the size of number that can have arbitrary persistence p. That is,* $\forall n > 1, P(n) = p, \qquad \exists m > n\colon\ P(m) = P(n) = p.$

Proof. Again the proof is by construction. Let $n = a_{x!}b_{(x-1)!}\dots c_{2!}d_{1!}$. Now shift all of the digits of n one place to the left and insert a 1 on the right; that is to say, construct $m = a_{(x+1)!}b_{x!}\dots c_{3!}d_{2!}1_{1!}$. Then $P(m) = P(n)$ since the product of the digits of m is the same as that of n, and, by repetition of the construction, it is possible to produce an arbitrarily large integer m' such that $P(m') = P(n)$. □

The simple observations made in this paper clearly only touch on the questions about persistence in factorial base numbers. Perhaps the two most obvious unanswered questions are

- Is it possible to improve the upper bound on the size of the smallest number with given persistence?

- How does excluding the zero digits from the calculaton of persistence, in line with the conjecture by Erdös Pàl [2], affect matters?

References

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