Multiplicative persistence base 10: some new null results

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Abstract

The main purpose of this paper is to present some new, null-results related to the multiplicative persistence of numbers in base 10. A secondary purpose is to introduce to the naïve, mathemaically unsophisticated reader, the main ideas behind the strategy used for searching for numbers with high multiplicative persistence.[1]

1 Introduction

Multiply together all the digits of a positive integer, n. Using the result, repeat the digit-multiplication process to obtain a new result. Continue until a single digit result is obtained. The number of steps, p, that it takes for n to be changed to the single digit end-point, is called the *multiplicative persistence* of n (Sloane, 1973).

The multiplicative persistence of an integer depends upon the base in which the integer is expressed.

2 Notation

I use the following notation to indicate that the digits of n_0 have been multiplied together (step 1, indicated as $\xrightarrow{1}$) to obtain n_1, and that the process has been repeated through steps $2 \ldots p$ until the single digit result n_p is obtained:

$$n_0 \xrightarrow{1} n_1 \xrightarrow{2} \quad \ldots \quad \xrightarrow{p-1} n_{p-1} \xrightarrow{p} n_p.$$

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3 Examples

Base 10

Commencing with the positive integer 3673, one proceeds as follows:

$$\begin{array}{rcllcl} 3673 & \xrightarrow{1} & 3 \times 6 \times 7 \times 3 & = & 378 \\ 378 & \xrightarrow{2} & 3 \times 7 \times 8 & = & 168 \\ 168 & \xrightarrow{3} & 1 \times 6 \times 8 & = & 48 \\ 48 & \xrightarrow{4} & 4 \times 8 & = & 32 \\ 32 & \xrightarrow{5} & 3 \times 2 & = & 6. \end{array}$$

The multiplicative persistence of 3673 is 5 because a single-digit result is reached after 5 steps.

Base 7

Commencing with the positive integer 320, which is represented in base 7 as 635_7, one proceeds as follows:

$$\begin{array}{rcllcl} 635_7 & \xrightarrow{1} & 6 \times 3 \times 5 & = & 156_7 \\ 156_7 & \xrightarrow{2} & 1 \times 5 \times 6 & = & 42_7 \\ 42_7 & \xrightarrow{3} & 4 \times 2 & = & 11_7 \\ 11_7 & \xrightarrow{4} & 1 \times 1 & = & 1_7. \end{array}$$

The multiplicative persistence of 635_7 is 4 because a single-digit result is reached after 4 steps.

4 Well known results

The multiplicative persistence of the first 100 positive integers is shown in Table 1 and the smallest integers with persistence from 1 to 11 are shown in Table 2. There are numerous entries relating to multiplicative persistence in the On-Line Encyclopedia of Integer Sequences (OEIS). Table 1 in this paper is the same as sequence A031346 at OEIS; Table 2 is the same as sequence A003001.

It is conjectured that there is no base-10 number with persistence greater than 11. However, no proof exists of the correctness of the conjecture (if

n	p	n	p	n	p	n	p
1	0	26	2	51	1	76	2
2	0	27	2	52	2	77	4
3	0	28	2	53	2	78	3
4	0	29	2	54	2	79	3
5	0	30	1	55	3	80	1
6	0	31	1	56	2	81	1
7	0	32	1	57	3	82	2
8	0	33	1	58	2	83	2
9	0	34	2	59	3	84	2
10	1	35	2	60	1	85	2
11	1	36	2	61	1	86	3
12	1	37	2	62	2	87	3
13	1	38	2	63	2	88	3
14	1	39	3	64	2	89	3
15	1	40	1	65	2	90	1
16	1	41	1	66	3	91	1
17	1	42	1	67	2	92	2
18	1	43	2	68	3	93	3
19	1	44	2	69	3	94	3
20	1	45	2	70	1	95	3
21	1	46	2	71	1	96	3
22	1	47	3	72	2	97	3
23	1	48	2	73	2	98	3
24	1	49	3	74	3	99	2
25	2	50	1	75	3	100	1

Table 1: The multiplicative persistence p of the positive integers $1 \leq n \leq 100$.

p	n	$2\uparrow$	$3\uparrow$	$5\uparrow$	$7\uparrow$
1	10				
2	25	1		1	
3	39		3		
4	77				2
5	679	1	3		1
6	6788	7	1		1
7	68889	10	3		
8	2677889	8	3		2
9	26888999	11	7		
10	3778888999	12	7		2
11	277777788888899	19	4		6

Table 2: The table shows the multiplicative persistence p of the positive integers $1 \leq n \leq 100$. The columns headed $2\uparrow$, $3\uparrow$, $5\uparrow$, $7\uparrow$ are explained in section 1 and indicate the frequency of the prime factors 2, 3, 5 and 7 in the decimal digits of n. For example, the individual digits of the number 679, shown in row five of the table, can be written as the product of primes as follows. $(2 \times 3)\ 7\ (3 \times 3)$. In that representation, the number 2 appears once (as shown in the column $2\uparrow$), 3 appears 3 times (as shown in the column $3\uparrow$), and 7 appears once (as shown in the column $7\uparrow$).

there were, we'd call it a theorem!) and occassionally new results about the search for a high persistence number are published.

5 Searching for high persistence numbers

A naïve approach to searching for integers with high persistence—that is, integers with a persistence greater than 11—is to determine the persistence of successive integers beginning with the number 1. A little bit of even trivial analysis, however, shows that the search can be improved dramatically. The easiest way to highlight the improvements that can be made is to begin with a few very simple observations and very elementary results from number theory.

5.1 Avoiding the digit 0

It is obvious that one not waste time checking any integer that contains the digit zero (0) since the product of the digits will produce a single digit after just one step. Example: $4709 \xrightarrow{1} 4 \times 7 \times 0 \times 9 = 0$.

5.2 Avoiding the digit 1

Consider the fact that if a number contains the digit 1 and has a multiplicative persistence of p, then there must be a smaller integer that also has persistence p. For example, if the digits of the integer are $d_1 d_2 d_3 \ldots 1 \ldots d_n$ (with the digit 1 possibly appearing in the first or last place rather than in the middle as shown here) then when we multiply the digits together we will obtain $d_1 \times d_2 \times \cdots \times 1 \times \cdots \times d_n$. But we would get the same result if we simply omitted the digit 1 wherever it appears.
Example: $911311111 \xrightarrow{1} 27 \xrightarrow{2} 14 \xrightarrow{3} 4$, which is the same sequence as for the much smaller number $93 \xrightarrow{1} 27 \xrightarrow{2} 14 \xrightarrow{3} 4$.

Conclusion: We need only search for high persistence numbers amongst those integers that do not contain the digit 1.

5.3 Increasing digits

If two different integers have identical digits (e.g., 329, 932) then they will clearly have the same multiplicative persistence because their digit products will converge at the first step.

Example: $329 \xrightarrow{1} 54 \xrightarrow{2} 20 \xrightarrow{3} 0$ and $932 \xrightarrow{1} 54 \xrightarrow{2} 20 \xrightarrow{3} 0$

Amongst all the integers with any given set of decimal digits (allowing for repeating digits), the integer that has the digits in ascending order will be the smallest.

Example: 3447 is smaller than any of 3474, 3744, 4347, 4374, 4437, 4473, 4734, 4743, 7344, 7434, or 7443. Conclusion: We need only search for high persistence numbers amongst those integers that have their digits in ascending order.

5.4 Representing digits as prime factors

Although we could limit our search for high-persistence numbers to only those numbers that

- did not contain either of the digits 0 or 1, and
- had digits that were in ascending order

there is one further significant simplification that can be made in the searching method.

Note that $479 \xrightarrow{1} 252 \xrightarrow{2} 20 \xrightarrow{3} 0$ and $667 \xrightarrow{1} 252 \xrightarrow{2} 20 \xrightarrow{3} 0$ both have a multiplicative persistence of 3. Note also that the digit products of the two numbers are identical (equal to 252) after step 1. The reason that the first step results are identical is that the prime-factorization of the *digits* of the two numbers 479 and 667 (rather than the prime factorization of the numbers themselves) is identical.

Rather than wasting time either (a) checking integers that have the same digit factorization as an integer that has already been checked, or (b) determining the prime factorization of the digits of the integers we are testing, it is easier simply to check the multiplicative persistence of integers that are formed only from prime digits. Neither 479 nor 667 would be checked in such a search; instead we would determine the persistence of the number that has the prime factorization of the *digits* of those two numbers 'spelled out' in its individual digits—namely, 22337. If our search leads us to the discovery that the number

$$22222222222222222223333777777$$

is the smallest with a multiplicative persistence of 11 amongst those that we have checked, then it is a simple matter to convert that number to the smallest number with a persistence of 11 by 'packing' the prime digits of the original number into as few digits as possible.

The algorithm for the packing is simplicity itself.

	d_1	d_2	d_3	$2\uparrow$	$3\uparrow$	$7\uparrow$
digits	4	7	9			
factorization	2×2	7	3×3	2	2	1
digits	6	6	7			
factorization	2×3	2×3	7	2	2	1

Table 3: On the rows labelled 'digits', the digits (d_1, d_2, d_3) of each of the two integers 479 and 667 are shown. Below them appears the prime factorization of those digits. The columns labelled $2\uparrow$, $3\uparrow$, and $7\uparrow$ show the frequency of occurrence of each of the prime numbers 2, 3 and 7 in the prime factorization of the digits. The important connexion between the various columns is that the product of the digits of 479 is equal to the product of the digits of 667 and is also equal to the product of the digits of the number $22337 \xrightarrow{1} 2^2 \times 3^2 \times 7^1 = 252$ where the exponents on 2, 3, and 7 correspond to the values shown in the columns labelled $2\uparrow$, $3\uparrow$, and $7\uparrow$.

- Replace any three 2s with 8
  2222222222222222222233337777777 $\rightsquigarrow$ 8888882333377777777
- Replace any two 3s with 9
  8888882333377777777 $\rightsquigarrow$ 88888829977777777
- Replace any 2 and 3 with 6—a rule that has no effect in the current instance
- Replace any two 2s with 4—another rule that has no effect in the current instance
- Sort the digits of the result into ascending order
  88888829977777777 $\rightsquigarrow$ 277 777 788 888 899.

Note that the digits 5 and 7 are left untouched.

5.5 Avoiding 2–5 combinations

The final rule for simplifying our search is just a combination of two previous rules.

If the prime factorization of the digits of an integer contains both the factor 2 and the factor 5, then the product of the digits will contain a factor of 10 and hence the product of the digits will end with a zero. When the digits of that product are multiplied together, the result will be zero. In other words, if the prime factorization of the digits of an integer contains both the factor 2 and the factor 5 then it will have a multiplicative persistence of exactly 2.

Since (by the simplification rule in section 5.4) we know to check only numbers with prime digits, we avoid integers whose digits include both a 2 and a 5.

6 Computer code

It is very easy to determine the persistence of a number using *MATHEMATICA*. In the code shown below the function `digitProduct` is used to calculate the product of the digits of integer n. The function `persistence` applies the function `digitProduct` to an integer n and to each successive digit product until a single digit (i.e., an answer ≤ 9) is produced.

```
digitProduct[n_] := Times @@ IntegerDigits[n]
persistence[n_] :=
   Length[NestWhileList[digitProduct, n, (# > 9) &]] - 1
```

The two functions are easily adapted to calculations in base b as follows:

```
digitProduct[n_, b_] := Times @@ IntegerDigits[n, b]
persistence[n_, b_] := Length[
   NestWhileList[digitProduct[#, b] &, n, (# > (b - 1)) &]
   ] - 1
```

I imagine that a very much faster program could be coded in C using the GNU Multiple Precision Arithmetic Library (GNU GMP) but I have not tried it. My cursory look at GMP suggested that one would need to use one of the library routines to convert each integer to a string in order to get the digit by digit representation, before using the the high-precision multiplication routines to calculate the digit product.

7 Summary of results

7.1 Background

In 2001, Phil Carmody reported that he had determined the persistence of

- all those integers that can be expressed as $2^k \times 3^l \times 7^m$ where $0 \le k + l + m \le 775$, and
- all those integers that can be expressed as $3^k \times 5^l \times 7^m$ where $0 \le k + l + m \le 775$.

Put another way, he checked the persistence of all the numbers that can be written as the digit sequence $2_k 2_{k-1} \dots 2_1 3_l 3_{l-1} \dots 3_1 7_m 7_{m-1} \dots 7_2 7_1$ with the same bounds on k, l, and m as above.

7.2 New (null) results

I used *MATHEMATICA* to replicate and extend Carmody's results. I determined the persistence of:

- all those numbers that can be expressed as $2^k 3^l 7^m$ where $0 \le k \le 1000, 0 \le l \le 1000, 0 \le m \le 1000$, and
- all those numbers that can be expressed as $3^k 5^l 7^m$ where $0 \le k \le 1000, 0 \le l \le 1000, 0 \le m \le 1000$.

Note that my approach to the bounds is different from that of Carmody.

The *MATHEMATICA* program that I used consisted, in essence, of just two statements, namely:

```
persistence237 = Flatten[
  Table[
    persistence[2^n2 * 3^n3 * 7^n7],
      {n2, 0, 1000}, {n3, 0, 1000}, {n7, 0, 1000}
  ]
]
```

and

```
persistence357 = Flatten[
  Table[
    persistence[3^n3 * 5^n5 * 7^n7],
      {n3, 0, 1000}, {n5, 0, 1000}, {n7, 0, 1000}
  ]
]
```

The function `persistence` is defined in section 6.

My main result is the null result. None of the integers that I tested had a multiplicative persistence ≥ 10 other than the two that were documented by Carmody (2001), namely $2^4 3^{20} 7^5$ and $2^{19} 3^4 7^6$. The result is equivalent to saying that I found no previously undiscovered integer with persistence ≥ 11 amongst all the numbers consisting of strings of up to one thousand 2s, one thousand 3s, and one thousand 7s, or strings of up to one thousand 3s, one thousand 5s, and one thousand 7s.

Other results

Clearly any integer constructed in the way described in the preceding part of this paper—namely an integer consisting only of instances of the digits 2, 3, and 7, or alternatively of instances of the digits 3, 5, and 7—has a multiplicative persistence of at least 2. The first step in the trajectory of such a number is guaranteed to be non-zero, in virtue of the starting integer containing no zero digits, and at least one further step is needed (taking the multiplicative persistence to at least 2) before a single-digit result might be obtained.

Almost all of the 1 000 000 000 numbers that I tested (products of powers of 2, 3, and 7, and products of powers of 3, 5, and 7) had a persistence of 2. Put another way, most numbers represented by strings of up to one thousand 2s, one thousand 3s, and one thousand 7s, or by strings of up to one thousand 3s, one thousand 5s, and one thousand 7s, have persistence 3. An ASCII file of the complete results (omitting those power products with persistence 2) is available for download. The file is surprisingly small.

8 Trivia

The largest number with a persistence ≥ 1 in the space that I searched was $2^{25} 3^{227} 7^{28}$, with a multiplicative persistence of 2. That, in turn, indicates that the the 280 digit number $7_{28} \ldots 7_1 3_{227} \ldots 3_1 2_{25} \ldots 2_1$ has a persistence of 3.

References

1. Carmody, P. (2001). OEIS A003001, and a “zero-length message”. Published online at *Archives of nmbrthry@listserv.nodak.edu*.

2. Sloane, N. J. A. (1973). The persistence of a number. *Journal of Recreational Mathematics,* 6, 97–98.