

# Multiplicative persistence base 10: some new null results

Mark R. Diamond

## Abstract

The main purpose of this paper is to present some new, null-results related to the multiplicative persistence of numbers in base 10. A secondary purpose is to introduce to the naïve, mathematically unsophisticated reader, the main ideas behind the strategy used for searching for numbers with high multiplicative persistence.<sup>1</sup>

## 1 Introduction

Multiply together all the digits of a positive integer,  $n$ . Using the result, repeat the digit-multiplication process to obtain a new result. Continue until a single digit result is obtained. The number of steps,  $p$ , that it takes for  $n$  to be changed to the single digit end-point, is called the *multiplicative persistence* of  $n$  (Sloane, 1973).

The multiplicative persistence of an integer depends upon the base in which the integer is expressed.

## 2 Notation

I use the following notation to indicate that the digits of  $n_0$  have been multiplied together (step 1, indicated as  $\xrightarrow{1}$ ) to obtain  $n_1$ , and that the process has been repeated through steps  $2 \dots p$  until the single digit result  $n_p$  is obtained:

$$n_0 \xrightarrow{1} n_1 \xrightarrow{2} \dots \xrightarrow{p-1} n_{p-1} \xrightarrow{p} n_p.$$

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### 3 Examples

#### Base 10

Commencing with the positive integer 3673, one proceeds as follows:

$$\begin{array}{rclcl} 3673 & \xrightarrow{1} & 3 \times 6 \times 7 \times 3 & = & 378 \\ 378 & \xrightarrow{2} & 3 \times 7 \times 8 & = & 168 \\ 168 & \xrightarrow{3} & 1 \times 6 \times 8 & = & 48 \\ 48 & \xrightarrow{4} & 4 \times 8 & = & 32 \\ 32 & \xrightarrow{5} & 3 \times 2 & = & 6. \end{array}$$

The multiplicative persistence of 3673 is 5 because a single-digit result is reached after 5 steps.

#### Base 7

Commencing with the positive integer 320, which is represented in base 7 as  $635_7$ , one proceeds as follows:

$$\begin{array}{rclcl} 635_7 & \xrightarrow{1} & 6 \times 3 \times 5 & = & 156_7 \\ 156_7 & \xrightarrow{2} & 1 \times 5 \times 6 & = & 42_7 \\ 42_7 & \xrightarrow{3} & 4 \times 2 & = & 11_7 \\ 11_7 & \xrightarrow{4} & 1 \times 1 & = & 1_7. \end{array}$$

The multiplicative persistence of  $635_7$  is 4 because a single-digit result is reached after 4 steps.

### 4 Well known results

The multiplicative persistence of the first 100 positive integers is shown in Table 1 and the smallest integers with persistence from 1 to 11 are shown in Table 2. There are numerous entries relating to multiplicative persistence in the [On-Line Encyclopedia of Integer Sequences \(OEIS\)](#). Table 1 in this paper is the same as sequence [A031346](#) at OEIS; Table 2 is the same as sequence [A003001](#).

It is conjectured that there is no base-10 number with persistence greater than 11. However, no proof exists of the correctness of the conjecture (if

$n$	$p$	$n$	$p$	$n$	$p$	$n$	$p$
1	0	26	2	51	1	76	2
2	0	27	2	52	2	77	4
3	0	28	2	53	2	78	3
4	0	29	2	54	2	79	3
5	0	30	1	55	3	80	1
6	0	31	1	56	2	81	1
7	0	32	1	57	3	82	2
8	0	33	1	58	2	83	2
9	0	34	2	59	3	84	2
10	1	35	2	60	1	85	2
11	1	36	2	61	1	86	3
12	1	37	2	62	2	87	3
13	1	38	2	63	2	88	3
14	1	39	3	64	2	89	3
15	1	40	1	65	2	90	1
16	1	41	1	66	3	91	1
17	1	42	1	67	2	92	2
18	1	43	2	68	3	93	3
19	1	44	2	69	3	94	3
20	1	45	2	70	1	95	3
21	1	46	2	71	1	96	3
22	1	47	3	72	2	97	3
23	1	48	2	73	2	98	3
24	1	49	3	74	3	99	2
25	2	50	1	75	3	100	1

**Table 1:** The multiplicative persistence  $p$  of the positive integers  $1 \leq n \leq 100$ .

$p$	$n$	$2 \uparrow$	$3 \uparrow$	$5 \uparrow$	$7 \uparrow$
1	10				
2	25	1		1	
3	39		3		
4	77				2
5	679	1	3		1
6	6788	7	1		1
7	68889	10	3		
8	2677889	8	3		2
9	26888999	11	7		
10	3778888999	12	7		2
11	277777788888899	19	4		6

**Table 2:** The table shows the multiplicative persistence  $p$  of the positive integers  $1 \leq n \leq 100$ . The columns headed  $2 \uparrow$ ,  $3 \uparrow$ ,  $5 \uparrow$ ,  $7 \uparrow$  are explained in section 1 and indicate the frequency of the prime factors 2, 3, 5 and 7 in the decimal digits of  $n$ . For example, the individual digits of the number 679, shown in row five of the table, can be written as the product of primes as follows.  $(2 \times 3) 7 (3 \times 3)$ . In that representation, the number 2 appears once (as shown in the column  $2 \uparrow$ ), 3 appears 3 times (as shown in the column  $3 \uparrow$ ), and 7 appears once (as shown in the column  $7 \uparrow$ ).

there were, we'd call it a theorem!) and occasionally new results about the search for a high persistence number are published.

## 5 Searching for high persistence numbers

A naïve approach to searching for integers with high persistence—that is, integers with a persistence greater than 11—is to determine the persistence of successive integers beginning with the number 1. A little bit of even trivial analysis, however, shows that the search can be improved dramatically. The easiest way to highlight the improvements that can be made is to begin with a few very simple observations and very elementary results from number theory.

### 5.1 Avoiding the digit 0

It is obvious that one not waste time checking any integer that contains the digit zero (0) since the product of the digits will produce a single digit after just one step. Example:  $4709 \xrightarrow{1} 4 \times 7 \times 0 \times 9 = 0$ .

### 5.2 Avoiding the digit 1

Consider the fact that if a number contains the digit 1 and has a multiplicative persistence of  $p$ , then there must be a smaller integer that also has persistence  $p$ . For example, if the digits of the integer are  $d_1 d_2 d_3 \dots 1 \dots d_n$  (with the digit 1 possibly appearing in the first or last place rather than in the middle as shown here) then when we multiply the digits together we will obtain  $d_1 \times d_2 \times \dots \times 1 \times \dots \times d_n$ . But we would get the same result if we simply omitted the digit 1 wherever it appears.

Example:  $911311111 \xrightarrow{1} 27 \xrightarrow{2} 14 \xrightarrow{3} 4$ , which is the same sequence as for the much smaller number  $93 \xrightarrow{1} 27 \xrightarrow{2} 14 \xrightarrow{3} 4$ .

Conclusion: We need only search for high persistence numbers amongst those integers that do not contain the digit 1.

### 5.3 Increasing digits

If two different integers have identical digits (e.g., 329, 932) then they will clearly have the same multiplicative persistence because their digit products will converge at the first step.

Example:  $329 \xrightarrow{1} 54 \xrightarrow{2} 20 \xrightarrow{3} 0$  and  $932 \xrightarrow{1} 54 \xrightarrow{2} 20 \xrightarrow{3} 0$





of 10 and hence the product of the digits will end with a zero. When the digits of that product are multiplied together, the result will be zero. In other words, if the prime factorization of the digits of an integer contains both the factor 2 and the factor 5 then it will have a multiplicative persistence of exactly 2.

Since (by the simplification rule in section 5.4) we know to check only numbers with prime digits, we avoid integers whose digits include both a 2 and a 5.

## 6 Computer code

It is very easy to determine the persistence of a number using *MATHEMATICA*. In the code shown below the function `digitProduct` is used to calculate the product of the digits of integer  $n$ . The function `persistence` applies the function `digitProduct` to an integer  $n$  and to each successive digit product until a single digit (i.e., an answer  $\leq 9$ ) is produced.

```
digitProduct[n_] := Times @@ IntegerDigits[n]
persistence[n_] :=
  Length[NestWhileList[digitProduct, n, (# > 9) &]] - 1
```

The two functions are easily adapted to calculations in base  $b$  as follows:

```
digitProduct[n_, b_] := Times @@ IntegerDigits[n, b]
persistence[n_, b_] := Length[
  NestWhileList[digitProduct[#, b] &, n, (# > (b - 1)) &]
] - 1
```

I imagine that a very much faster program could be coded in C using the [GNU Multiple Precision Arithmetic Library \(GNU GMP\)](#) but I have not tried it. My cursory look at GMP suggested that one would need to use one of the library routines to convert each integer to a string in order to get the digit by digit representation, before using the the high-precision multiplication routines to calculate the digit product.

## 7 Summary of results

### 7.1 Background

In 2001, Phil Carmody reported that he had determined the persistence of



- all those integers that can be expressed as  $2^k \times 3^l \times 7^m$  where  $0 \leq k + l + m \leq 775$ , and
- all those integers that can be expressed as  $3^k \times 5^l \times 7^m$  where  $0 \leq k + l + m \leq 775$ .

Put another way, he checked the persistence of all the numbers that can be written as the digit sequence  $2_k 2_{k-1} \dots 2_1 3_l 3_{l-1} \dots 3_1 7_m 7_{m-1} \dots 7_2 7_1$  with the same bounds on  $k$ ,  $l$ , and  $m$  as above.

## 7.2 New (null) results

I used *MATHEMATICA* to replicate and extend Carmody's results. I determined the persistence of:

- all those numbers that can be expressed as  $2^k 3^l 7^m$  where  $0 \leq k \leq 1000, 0 \leq l \leq 1000, 0 \leq m \leq 1000$ , and
- all those numbers that can be expressed as  $3^k 5^l 7^m$  where  $0 \leq k \leq 1000, 0 \leq l \leq 1000, 0 \leq m \leq 1000$ .

Note that my approach to the bounds is different from that of Carmody.

The *MATHEMATICA* program that I used consisted, in essence, of just two statements, namely:

```
persistence237 = Flatten[
  Table[
    persistence[2^n2 * 3^n3 * 7^n7],
    {n2, 0, 1000}, {n3, 0, 1000}, {n7, 0, 1000}
  ]
]
```

and

```
persistence357 = Flatten[
  Table[
    persistence[3^n3 * 5^n5 * 7^n7],
    {n3, 0, 1000}, {n5, 0, 1000}, {n7, 0, 1000}
  ]
]
```

The function `persistence` is defined in section 6.

My main result is the null result. None of the integers that I tested had a multiplicative persistence  $\geq 10$  other than the two that were documented

by Carmody (2001), namely  $2^43^{20}7^5$  and  $2^{19}3^47^6$ . The result is equivalent to saying that I found no previously undiscovered integer with persistence  $\geq 11$  amongst all the numbers consisting of strings of up to one thousand 2s, one thousand 3s, and one thousand 7s, or strings of up to one thousand 3s, one thousand 5s, and one thousand 7s.

## Other results

Clearly any integer constructed in the way described in the preceding part of this paper—namely an integer consisting only of instances of the digits 2, 3, and 7, or alternatively of instances of the digits 3, 5, and 7—has a multiplicative persistence of at least 2. The first step in the trajectory of such a number is guaranteed to be non-zero, in virtue of the starting integer containing no zero digits, and at least one further step is needed (taking the multiplicative persistence to at least 2) before a single-digit result might be obtained.

Almost all of the 1 000 000 000 numbers that I tested (products of powers of 2, 3, and 7, and products of powers of 3, 5, and 7) had a persistence of 2. Put another way, most numbers represented by strings of up to one thousand 2s, one thousand 3s, and one thousand 7s, or by strings of up to one thousand 3s, one thousand 5s, and one thousand 7s, have persistence 3. An ASCII file of the complete results (omitting those power products with persistence 2) is available for download. The file is surprisingly small.

## 8 Trivia

The largest number with a persistence  $\geq 1$  in the space that I searched was  $2^{25}3^{227}7^{28}$ , with a multiplicative persistence of 2. That, in turn, indicates that the the 280 digit number  $7_{28} \dots 7_1 3_{227} \dots 3_1 2_{25} \dots 2_1$  has a persistence of 3.

## References

1. Carmody, P. (2001). OEIS A003001, and a “zero-length message”. Published online at [Archives of nmbrthry@listserv.nodak.edu](mailto:Archives_of_nmbrthry@listserv.nodak.edu).
2. Sloane, N. J. A. (1973). The persistence of a number. *Journal of Recreational Mathematics*, 6, 97–98.