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FACTORIAL-BASE REPRESENTATIONS AND AUTOMORPHIC NUMBERS

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Readers may enjoy investigating aspects of factorial base representations and, in particular, the automorphic squares to that base. Since automorphic squares and factorial base representations bear no inherent relationship to one another, we begin with a brief description of each.

Factorial base representations are simply an example, albeit an interesting one, of the variety of multiple base representation systems which include the Imperial weights, measures, and coinage system and the radial degree-minute-second system [1]. In factorial base, numbers are represented as the sum of multiples of factorials and their reciprocals [2, 3]. The first digit to the left of the fractional point represents multiples of 1!, the second digit to the left represents multiples of 2!, and so forth. Similarly, to the right of the fractional point are the multiples of 1/2!, 1/3!, and so on. Thus,

$$11_{10} = (1 \times 3!) + (2 \times 2!) + (1 \times 1!) = 121_F$$

$$106.5_{10} = (4 \times 4!) + (1 \times 3!) + (2 \times 2!) + (0 \times 1!) + \frac{1}{2!} = 4120 \cdot 1_F$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots = 10 \cdot 11\overline{1}_F$$

Factorial base representations provide a convenient method of performing multiple precision arithmetic on a computer since it is possible to represent any rational as a factorial base number of finite length. This is easy to see if we let *p/q* be a

Table 1. Integers Less Than 30000 Which Are Automorphic Squares in Factorial Base Representation.[a]

n	n_F	n_F^2
1	1	1
3	11	111
4	20	220
9	111	3111
16	220	20220
25	1001	51001
40	1220	211220
81	3111	1203111
96	4000	1544000
105	4111	2114111
145	11001	4111001
225	14111	12014111
496	40220	60540220
576	44000	81544000
640	51220	111151220
721	100001	137100001
945	114111	241114111
1296	144000	455144000
2016	244000	1116244000
2080	251220	1182251220
2241	303111	1374303111
2800	351220	2153351220
2961	403111	2413403111
3025	411001	2517411001
3745	511001	3856511001
4096	540220	4620540220
4320	600000	5136600000
4816	640220	6381640220
5761	1100001	9141100001
8065	1411001	16921411001
9856	1640220	24761640220
12160	2251220	37742251220
13825	2511001	48662511001
17920	3351220	80483351220
22401	4303111	106274303111
26496	5144000	156455144000
28161	5403111	179535403111

[a] The table shows both the base 10 and factorial-base representation of n together with the factorial base representation of n^2.

Table 2. Base 10 Representations of Integers between 30000 and 4000000 Which Are Automorphic Squares in Factorial Base Representation

n	*n*	*n*
30465	207361	1223425
32256	213760	1374976
34560	228096	1741825
36225	232065	1886976
58240	286336	2253825
72576	290305	2405376
76545	304541	2598400
130816	512001	2772225
134785	518400	3110401
149121	856576	3116800
155220	1030401	3628801

fraction in reduced form, so that $(p,q) = 1$. The fraction p/q can then be represented as some multiple of $(k!)^{-1}$ where k is the smallest integer such that $q \mid k!$. The representation of p/q will therefore require at most $k - 1$ digits to the right of the fractional point.

As many readers will recall, the automorphic numbers are those which, when squared, end in the same digits as themselves [4, 5]. As deGuerre and Fairburn put it, "... a number is called automorphic in the scale of notation with base B if all its powers end in the same digits in this scale of notation." Thus, to base 10, for example, $(76_{10})^2 = 5776_{10}$ and to base 6, $(213_6)^2 = 50213_6$.

We ran a computer program based in part on Hegeman's code [3], to determine all those numbers less than 4000000 whose factorial base representation has a similar automorphic property. For numbers less than 30000, detailed results are presented in Table 1 while Table 2 gives only the base 10 representation for values between 30000 and 4000000. Some of the more obvious questions related to the factorial base automorphic squares are:

1. Are there an infinite number of them?
2. If so, how common are they? and
3. Is there a method for finding the successive factorial base automorphic squares other than by exhaustive search?

In the same way as Hunter extended the notion of automorphic number to powers of three in single base representations [6], so too can the idea be extended to powers greater than 2 with factorial base representation, but we leave this to the reader.