Strategy in a Television Game-Show

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1 Background

Imagine the following guessing game ... The game show host announces that she has selected an integer in the interval $[l, u]$ and that you and I are to take it in turns to guess the integer. After each guess, the host will announce whether the guess is right or wrong. The first player correctly to guess the number will win a trip to the South Pacific. Clearly, it is simply a matter of chance whether it is you or I who travels to sunnier climes.

Now imagine a slight variation of the game. Instead of merely announcing whether a guess was right or wrong, the game-show host additionally informs us whether the guess was too high or too low. This variation is, in fact, the essence of a game-show called "The Price is Right" which is screened on Australian television, and some further details of which appear in [2].

At first sight, the alteration makes little change to the substance of the original guessing game. Indeed, this was the view expressed by every one of half a dozen colleagues we spoke to in an informal poll. Needless to say, there would be no paper if this impression were correct!

If $[l, u]$ is the interval within which the parcel's true value is known to lie prior to some turn (designated turn A), then it is easy to see that the probability that the next contestant (B) wins on the following turn is $\frac{1}{u-l+1}$. The interesting question then arises as to the strategy which maximizes A's overall chance of winning and minimizes the chance that the other contestant (B) will win.

2 Strategy

It is not enough to study the impact of a player's choice on the next stage of the game. One needs to take into consideration the rest of the game. Analyzing the game this way, one finds that the following is an optimal strategy[1].

[1] We are indebted to Professor Dov Samet for the following analysis. He corrected an earlier, insufficiently general, strategy, that we had devised.

2.1 The rules

1. When faced with an even number of numbers, choose any of them.

2. When the number of numbers is odd, choose one that has on both sides of it an even number of numbers.

2.2 Probabilities and proof

The rules can be shown to give the first player a winning chance of $\frac{1}{2}$ when the initial number of numbers is even, and a chance of $\frac{1}{2} + \frac{1}{2n}$ when the size of the interval, $n = u - l + 1$, is odd.

Proof. The proof is by induction on n. The cases $n = 2$ and $n = 3$ are trivially as follows:

If $n = 2$, and we use c to represent Player 1's choice, then she can choose either $c = 1$ or $c = 2$. It makes no difference which is chosen; each has a probability of $\frac{1}{2}$ of being correct. If the choice is correct then Player 1 wins immediately with probability $\frac{1}{2}$, or else the guess is incorrect and Player 2, who is left with no effective choice, wins on the following turn.

If $n = 3$, then Player 1 can choose either end-point of the interval—that is $c = 1$ or $c = 3$—leaving an even numbered interval on either side of c. One of those even intervals will of necessity be of size 0. Player 1 will win immediately with probability $\frac{1}{n} = \frac{1}{3}$, will lose on the subsequent turn with probability $\frac{1}{2} \cdot \frac{n-1}{n}$, or win on the turn after that with probability $\frac{1}{2} \cdot \frac{n-1}{n}$. Thus the total probability of Player 1 winning by choosing an end-point from the interval $[1, 3]$ is $\frac{1}{3} + (1 - \frac{1}{3})\frac{1}{2} = \frac{2}{3}$.

If Player 1 were to choose $c = 2$, contrary to the proposed winning strategy, she would leave an odd-sized interval on either side of c and would win immediately with probability $\frac{1}{n} = \frac{1}{3}$ or lose to Player 2 on the next turn. Thus the total probability of Player 1 winning by playing contrary to the proposed strategy in the interval $[1, 3]$ is only $\frac{1}{3}$.

For $n = 2$ or $n = 3$ our hypothesis is therefore correct.

Suppose now that $n \equiv 1 \bmod 2$, $n > 3$. Player 1 chooses a number c that leaves even intervals on both sides, say of sizes $k = n - c$ and $n - k - 1$. By the induction hypothesis, Player 2, when it is her turn to play, will be able to guarantee a winning chance of $1/2$ in each one of these intervals. Therefore $1 - \frac{1}{2} = \frac{1}{2}$ is also what Player 1 can guarantee for herself in each one of the intervals. The components of Player 1's overall chance of winning are

- the chance of winning immediately, namely $\frac{1}{n}$;

- the chance of winning on a later turn, $\frac{1}{2}$ multiplied by the probability that the winning value will lie in the interval of size k; and finally

- the chance of winning on a later turn, $\frac{1}{2}$ multiplied by the probability that the winning value will lie in the interval of size $n-k-1$.

That is to say, Player 1's total chance of winning is

$$\frac{1}{n} + \left(\frac{1}{2} \times \frac{k}{n}\right) + \left(\frac{1}{2} \times \frac{n-k-1}{n}\right) = \frac{1}{2} + \frac{1}{2n}. \tag{1}$$

Suppose $n \equiv 0 \bmod 2$. Player 1 chooses a number that leaves one odd interval of size k and one even interval of size $n-k-1$.

By the induction hypothesis, Player 2, when it is her turn to play, will be able to guarantee a winning chance of $\frac{1}{2}$ in the even $(n-k-1)$ sized interval, and of $\frac{1}{2} + \frac{1}{2k}$ in the odd sized interval.

Player 1 can guarantee herself a winning chance of $1 - \frac{1}{2} = \frac{1}{2}$ in the even interval and $1 - (\frac{1}{2} + \frac{1}{2k}) = \frac{1}{2} + \frac{1}{2k}$ in the odd one.

Muliplying and summing the probabilities as in the earlier list, one thus finds that the total chance of Player 1 winning is

$$\frac{1}{n} + \frac{k}{n}\left(\frac{1}{2} - \frac{1}{2k}\right) + \frac{1}{2}\left(\frac{n-k-1}{n}\right) = \frac{1}{2}. \tag{2}$$

□

3 *EasyOptimal* — a strategy for the innumerate

For the many people who would not trust themselves to determine whether a single interval was or was not even sized—let alone whether the interval on either side of a choice-number was even sized—we have a simple rule.

Choose a number from one of the ends of the interval.

The rule is easy to remember and will result in optimal play. If the interval is of even length, then choosing a number from one of the ends conforms to Rule 1 above. Similarly, if the interval is of odd length, then choosing one of the end numbers will reduce the interval length by one, making the length even on one side, and zero (also even) on the other—a choice which conforms to Rule 2.

4 Discussion

From the foregoing analysis, it is clear that optimal play gives a slight but important advantage over random strategy. Furthermore, knowledge of the strategy becomes potentially more important as the game progresses. If, for example, you are the player who is initially presented with an even sized interval, then there is no choice available to you that will increase your winning chance beyond $\frac{1}{2}$. However, if at some point in the game your opponent makes a suboptimal play, then you immediately gain an advantage. Furthermore, the extent of your change of fortune will be *relatively* greater if your opponent's slip occurs later in the game than if it occurs in the early stages.

The question arises as to why it is that contestants do not play optimally. We would suggest two reasons. First, the game is not iterated, so no single contestant ever has the opportunity to compare different strategies of play. Second, no contestant has ever used the optimal strategy, and thus no future player will have had the opportunity for vicarious learning [1]. Finally, it is amusing to consider what would happen were one or both of the two contestants to employ our *EasyOptimal* strategy. With one player using this strategy, a game would, on average, last twice as long as currently, and the audience would probably become bored. Were both players ever to rely on *EasyOptimal* the game could be expected to last 50 turns, and it would probably bring about its total demise as a television entertainment.

References

[1] A. Bandura. *Social learning theory.* Prentice Hall, Englewood Cliffs, New Jersey, 1977.

[2] D. D. Reidpath and M. R. Diamond. A non-experimental demonstration of anchoring bias. *Psychological Reports*, 76, 1995.